An Endogenous Gridpoint Method for Distributional Dynamics

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April 2024

Motivation

- Heterogeneous agent models have at their core the evolution of the distribution of agents.
  (Krusell and Smith, 1998, Reiter, 2009, Den Haan et al., 2010, Bayer and Luetticke, 2020, Boppart et al., 2018, Auclert et al., 2021, ...)
- Distributional dynamics usually computed with "histogram method".
  (Young, 2010)
- This method is computationally intensive and fails to capture how the distribution responds to aggregate risk. (Bhandari et al., 2023)

This Paper: Novel endogenous gridpoint method to model distributional dynamics

- ***DEGM*** (distributional endogenous gridpoint method):
  - It's fast: No integration
  - It's simple: Histogram representation
  - It's shape preserving: Efficient approximation
  - It's nonlinear: Captures aggregate risk
- Method yields large gains in numerical accuracy for all orders of solutions.

- Application: Third-order solution of heterogeneous agent model with aggregate investment risk.
- Nonlinear distributional dynamics imply higher wealth inequality.

Problem and Method

Setup

- Consider an economy in discrete time with a distribution of agents (of mass 1) over two variables a and y.
- y follows an exogenous discrete Markov process with transition probability matrix Π and set of states $\{\mathcal{Y}_j\}$.
- The continuous endogenous variable a is determined by the agent's policy function $a^*(a, y)$, which we assume to be strictly monotone in a.
- The cumulative joint distribution (in a) at time t is given by $F_t(a, y) := P(x \leq a, z = y)$, where $f_t(a, y)$ is the density (continuous along the a dimension, discrete along the y dimension).

Problem of Distributional Dynamics

The evolution of the distribution F is given by the time discrete Kolmogorov forward equation:

$$F_{t+1}(a', y') = \sum_j \int_{\{x|a' \geq a^*(x, \mathcal{Y}_j)\}} f_t(x, \mathcal{Y}_j) dx \quad \Pi(\mathcal{Y}_j, y'). \tag{1}$$

- A brute force approach requires approximation of the integral
- Krusell and Smith (1998) use Monte Carlo methods to solve the Eq. (1)

Histogram Approach

To avoid this, Young (2010) suggests replacing the continuous distribution in a with a discrete counterpart, what is commonly referred to as histogram method.

Histogram method:
Density given by the vector $\hat{f}$ with transition matrix $\mathbf{A}^*$ (policy function $a^*(a, y)$ as lotteries) such that:

$$\hat{f}_{t+1} = \hat{f}_t \mathbf{A}^*. \tag{2}$$

Lottery weights determined by relative distance of a^* to gridpoints

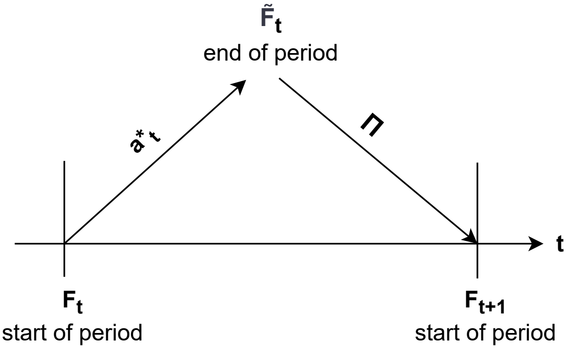
Our Method: Timing

First define

$$\tilde{F}_t(a', \mathcal{Y}_j) = \int_{\{x|a' \geq a^*(x, \mathcal{Y}_j)\}} f_t(x, \mathcal{Y}_j) dx. \tag{3}$$

such that

$$F_{t+1}(a', y') = \sum_j \quad \tilde{F}_t(a', \mathcal{Y}_j) \quad \Pi(\mathcal{Y}_j, y'). \tag{4}$$



Our Method: Leverage Monotonicity

- How to evaluate $\tilde{F}_t(a', \mathcal{Y}_j)$?
- Consider the endogenous gridpoints $a' = a^*(a, \mathcal{Y}_j)$.
- Use that $a^*(\cdot, \mathcal{Y}_j)$ is strictly monotone everywhere and thus invertible.

Our Method: Leverage Monotonicity

- For these points $a' = a^*(a, \mathcal{Y}_j)$, the set over which we integrate simplifies to:

$$\{x | a' \geq a^*(x, \mathcal{Y}_j)\} = \{x | a^*(a, \mathcal{Y}_j) \geq a^*(x, \mathcal{Y}_j)\} = \{x | a \geq x\}$$

(where the last equation results from the invertibility of a^*)

- This, together with the definition of F_t, again implies that

$$\boxed{\underbrace{\tilde{F}_t(a^*(a, \mathcal{Y}_j), \mathcal{Y}_j)}_{\text{endogenous grid}} = \underbrace{F_t(a, \mathcal{Y}_j)}_{\text{exogenous grid}}}$$

Illustration 1: Beginning of period CDF $\mathbf{F}_t(1:3, j)$

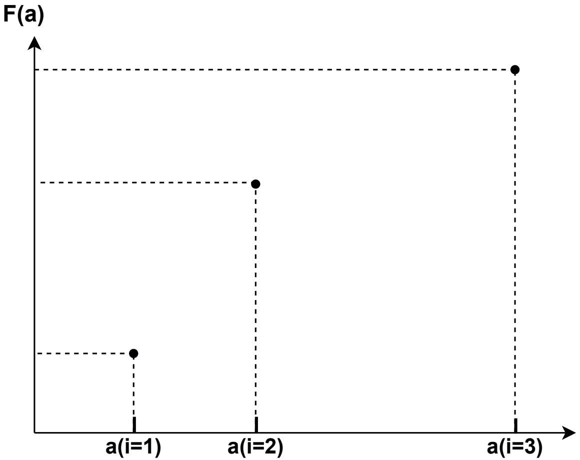


Illustration 2: Graph $\left\{\left(\mathcal{A}^*_{1:3,j}, \mathbf{F}_t(1:3,j)\right)\right\}$

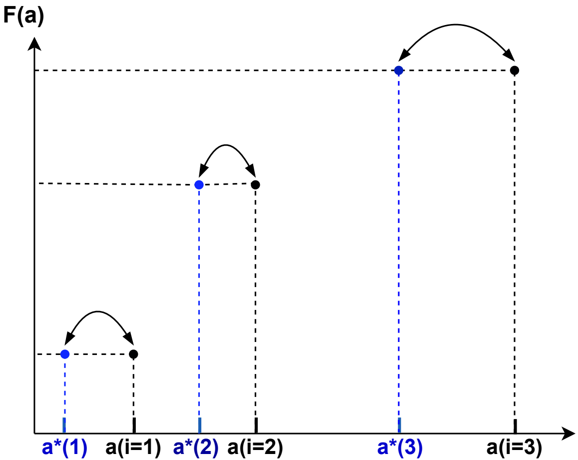


Illustration 3: Construction of interpolant $\hat{\tilde{F}}_t^j$

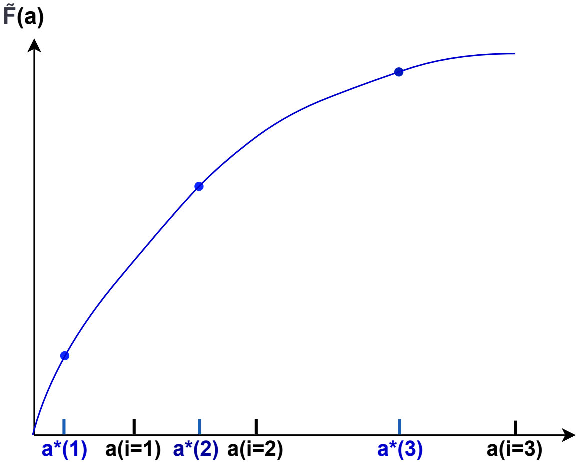
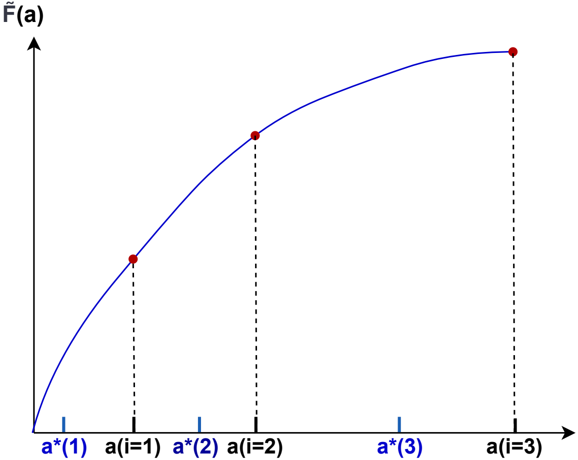


Illustration 4: Evaluation of interpolant $\hat{\tilde{\mathbf{F}}}_t(1:3, j)$



Algorithm in a Nutshell

Start with the cumulative joint distribution (in a) $\mathbf{F}_t = [F_t(\mathcal{A}_i, \mathcal{Y}_j)]_i^j$.

1. For each exogenous state with index j, $y = \mathcal{Y}_j$, **create the interpolant** $\hat{\tilde{F}}_t^j$.
2. Loop through all i, j to **evaluate the interpolant** to calculate:

$$\hat{\tilde{\mathbf{F}}}_t(i,j) = \begin{cases} 0 & \text{if } \mathcal{A}_i < \min\left\{\mathcal{A}^*_{i,j}\right\} \\ \mathbf{F}_t(end, j) & \text{if } \mathcal{A}_i > \max\left\{\mathcal{A}^*_{i,j}\right\} \\ \hat{\tilde{F}}_t^j(\mathcal{A}_i) & \text{else} \end{cases}$$

   This yields the CDF in a on the fixed grid $\{\mathcal{A}_i\}$ prior to the exogenous Markov transitions.

3. Apply the **exogenous** Markov **transition** matrix Π to obtain $\mathbf{F}_{t+1}$ as:

$$\mathbf{F}_{t+1} = \hat{\tilde{\mathbf{F}}}_t \Pi'$$

Implementation

- Since cumulative distribution functions are monotone, it is advisable to use an interpolation routine that preserves monotonicity.
- Both linear interpolation and piecewise cubic hermitian splines have this property.
- We use the latter to preserve the shape and differentiability of the distribution function.

Nonlinear Distributional Dynamics

- For simplicity, consider the discretized Kolmogorov forward equation.
- Density is given by the vector $\hat{f}$ with transition matrix $\mathbf{A}^*$ such that:

$$\hat{f}_{t+1} = \hat{f}_t \mathbf{A}^*. \tag{5}$$

- Consider a generic perturbation D_t, for example, aggregate shocks or changes in the mean of the distribution.

Nonlinear Distributional Dynamics

- The second order derivative of the transition matrix $\mathbf{A}^*(k,l)$ is given by

$$\frac{\partial \mathbf{A}^*(k,l)}{\partial a_k^*}\frac{\partial^2 a_k^*}{\partial D_t^2} + \frac{\partial^2 \mathbf{A}^*(k,l)}{\partial {a^*}_k^2}\left[\frac{\partial a_k^*}{\partial D_t}\right]^2, \tag{6}$$

  where a_k^* denotes the optimal policy at gridpoint a_k.

- The first effect captures the direct nonlinearity of the policy function.
- The second effect reflects that the Kolmogorov forward equation is in principle nonlinear in policies.

Nonlinear Distributional Dynamics of the Histogram Method

- The histogram method constructs $\mathbf{A}^*$, ignoring the exogenous state transitions for simplicity, as

$$\mathbf{A}^*(k,l) = \begin{cases} 1 - \frac{a_k^* - \mathcal{A}_l}{\mathcal{A}_{l+1} - \mathcal{A}_l} & \text{if } a_k^* \in [\mathcal{A}_l, \mathcal{A}_{l+1}) \\ \frac{a_k^* - \mathcal{A}_{l-1}}{\mathcal{A}_l - \mathcal{A}_{l-1}} & \text{if } a_k^* \in [\mathcal{A}_{l-1}, \mathcal{A}_l) \\ 0 & \text{else} \end{cases} \tag{7}$$

- This transition matrix is linear in a^*.

- Therefore $\frac{\partial^2}{\partial {a^*}_k^2}\mathbf{A}^*(k,l) = 0.$

Nonlinear Distributional Dynamics of DEGM

- The second-order derivative of our interpolant $\hat{\tilde{F}}_t^j(\mathcal{A}_i)$, has the general form:

$$\frac{\partial^2 \hat{\tilde{\mathbf{F}}}_t(i,j)}{\partial D_t^2} = \frac{\partial \hat{\tilde{\mathbf{F}}}_t(i,j)}{\partial \mathcal{A}^*_{\cdot j}} \frac{\partial^2 \mathcal{A}^*_{\cdot j}}{\partial D_t^2} + \left[\frac{\partial \mathcal{A}^*_{\cdot j}}{\partial D_t}\right]' \frac{\partial^2 \hat{\tilde{\mathbf{F}}}_t(i,j)}{\partial {\mathcal{A}^*_{\cdot j}}^2} \left[\frac{\partial \mathcal{A}^*_{\cdot j}}{\partial D_t}\right] \tag{8}$$

- Unlike the histogram method, the second term is nonzero because $\mathcal{A}^*_{\cdot j}$ are the vectors of the interpolation nodes (and the derivatives are vector-valued).

- Therefore, the Hessian $\frac{\partial^2 \hat{\tilde{\mathbf{F}}}_t(i,j)}{\partial {\mathcal{A}^*_{\cdot j}}^2}$ is generally nonzero.

Implementation

- As described in Bhandari et al. (2023), the second term in Equation 8 reflects second-order responses of the distributional dynamics to first-order changes in the optimal policy.
- Typically the continuous distribution has *curvature* at these pre-images, $\mathcal{A}^*_{\cdot j}$, hence approximation of $\frac{\partial^2 \hat{\tilde{\mathbf{F}}}_t(i,j)}{\partial \mathcal{A}^{*\,2}_{\cdot j}}$ requires a shape-preserving interpolation method.
- In the paper we extend this analysis to third order.

Application

Application: Model

Heterogeneous households as in Aiyagari (1994):

$$max_{\{c_t,k_{t+1}\}_{t=0}^{\infty}} \mathbb{E} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

$$\text{s.t. } c_t + k_{t+1} = (1 + r_t - \delta_t) k_t + h_t w_t N$$

$$k_{t+1} \geq 0$$

where

- k_t households savings
- h_t idiosyncratic productivity state, follows Markov process $\Pi(h, h')$
- $Z_t = \{F_t, \delta_t\}$ are aggregate states: cumulative distribution F and shocks δ_t

Results in policy function $k_{t+1} = k^*(k_t, h_t \mid Z_t)$

- Solve on grid for $(k, h) \in \mathcal{G}^k \text{x} \mathcal{G}^h$ with $|\mathcal{G}^k| = n_k$

Application: Model

Firm problem:

$$Y_t = K_t^{\alpha} N^{1-\alpha}$$
$$w_t = (1-\alpha)\left(\frac{K}{N}\right)^{\alpha}$$
$$r_t = \alpha\left(\frac{N}{K_t}\right)^{(1-\alpha)}$$

Equilibrium:
Households and firms optimize given prices such that the capital market clears,
$K_t = E_t[k]$, using $b \cdot F(b) - a \cdot F(a) - \int_a^b F(x)dx$

Capital depreciation shock

- δ_t follows *right-skewed* distribution and implies aggregate investment risk.
- Mimicks (binomial) disaster risk, as in Levintal (2017).
- Calibrate size and probability of “disaster” as in Barro (2006).

Calibration

Standard calibration following Den Haan et al. (2010)

Parameters	Description	Value
β	Discount factor	0.99
γ	Relative risk aversion	2
α	Capital share	0.36
δ	Depreciation rate	0.025
ρ_δ	Persistence of disaster	0.0
σ_δ	Second moment disaster	0.005
τ_δ	Third moment disaster	0.012
n_k	gridpoints for k	50-500
n_h	gridpoints for h	2

Solving Stationary Distributions

We perform two exercises:

- First, we isolate the quality of the approximation of the distribution by keeping prices and optimal policies fixed at the benchmark solution ($n_k = 500$ with *DEGM*).
- We select a subset of gridpoints from this solution and iterate on the distribution until convergence for both the histogram method and our *DEGM*. (We use piecewise cubic hermitian splines to interpolate the cumulative distribution function.)

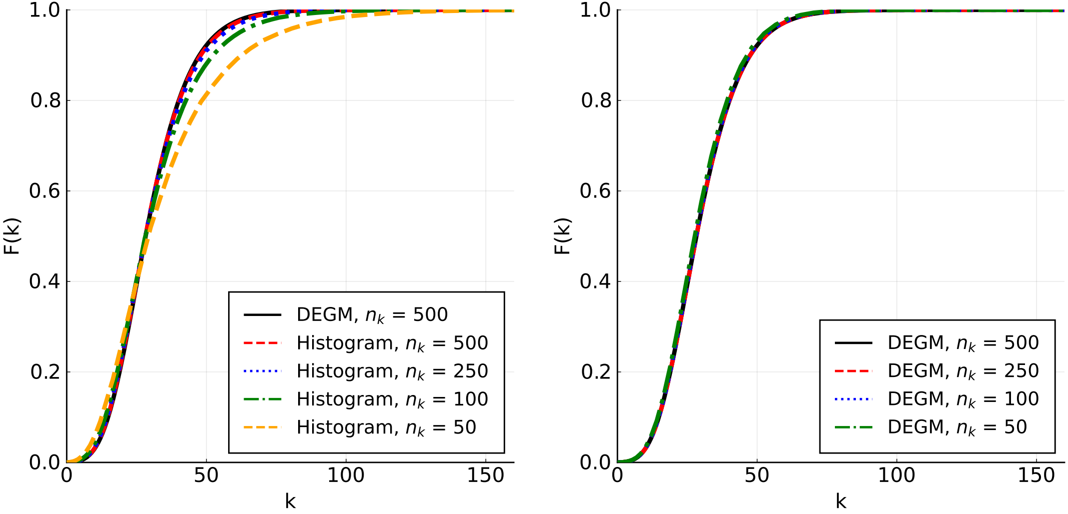
- Second, we solve for the stationary equilibrium, including prices and policies, which more closely resembles the actual use case.

Solving Stationary Distributions

	Histogram			DEGM		
$n_k=$	**50**	**100**	**250**	**50**	**100**	**250**
Capital stock	14.33	4.10	1.00	-2.33	-0.59	-0.04
Wealth gini	33.74	13.74	4.79	-2.61	-1.29	-0.25
Time	0.00	0.01	0.02	0.06	0.10	0.18

Note: Values represent percent deviations of solution with n_k gridpoints from the converged solution (DEGM and $n_k = 500$) (given the prices and policies under the converged solution). Time refers to the computation time in seconds that it takes to solve for the stationary distribution on a laptop with 16-core, 3.3GHz CPU.

Comparison of CDFs



Solving Stationary Equilibrium

	Histogram			DEGM		
$n_k=$	**50**	**100**	**250**	**50**	**100**	**250**
Capital stock	0.33	0.11	0.03	-0.09	-0.02	-0.00
Wealth gini	27.40	11.71	4.29	-0.86	-0.95	-0.26
Time	0.33	0.71	2.40	1.00	1.75	4.16

Note: Values represent percent deviations of solution with n_k gridpoints from the converged solution (DEGM and $n_k = 500$). Time refers to the computation time in seconds that it takes to solve for the stationary equilibrium on a laptop with 16-core, 3.3GHz CPU.

Comparison of Histogram Method and DEGM

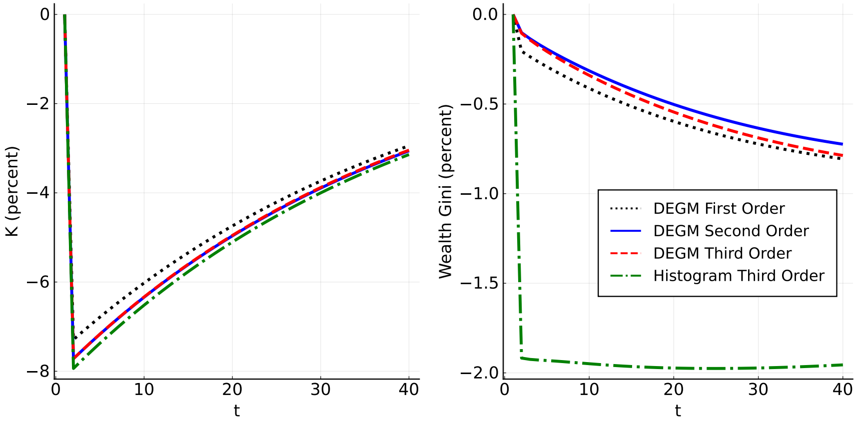
- *DEGM* converges to the “true” distribution much faster in the number of gridpoints, especially for cross-sectional moments.
- For a given number of gridpoints, the histogram method is faster in terms of computational time, mainly because it does not require iterations to update the distribution.
- However, for a given accuracy, our method is faster when solving for the stationary equilibrium.

Solving Distributional Dynamics with Aggregate Risk

- Extend state-space perturbation techniques from Bayer and Luetticke (2020) to higher orders (Levintal, 2017; Andreasen et al., 2018).
- Analyze differences in ergodic moments and distributions in second and third order across both methods.
- Analyze responses to capital destruction shock δ_t and their state dependence.

Impulse Responses

Response to 7.5 p.p. capital depreciation shock $(\nu_1 = 15\sigma_\delta)$, perturbations evaluated at steady state



Impulse Responses: Nonlinearities

- The first-order solution slightly understates the decline in aggregate capital and overstates the decline in the Gini coefficient of wealth in response to the capital depreciation shock.
- Distributional dynamics in this model are nonlinear with respect to aggregate shocks.
- However, the feedback from inequality to equilibrium prices is modest.
- The histogram method overstates the decline in the capital stock and the Gini coefficient.

First-Order Perturbation Solution

Accumulated differences of the impulse responses following a 7.5 p.p. shock to δ.

	Histogram			DEGM		
n_k=	**50**	**100**	**250**	**50**	**100**	**250**
Capital stock	5.00	2.40	0.91	1.13	0.23	0.01
Wealth gini	17.93	11.40	2.76	0.60	0.34	0.21
Time	0.01	0.03	0.30	0.01	0.03	0.32

Note: Values represent accumulated (absolute) differences in the responses following a 15-std. shock to δ (over 300 periods) from the first-order solution with n_k gridpoints to the converged solution (n_k=500 and DEGM). Values are given as percent differences to steady state.

Second-Order Perturbation Solution

Accumulated differences of the impulse responses following a 7.5 p.p. shock to δ.

	Histogram			DEGM		
$n_k =$	**50**	**100**	**250**	**50**	**100**	**250**
Capital stock	11.13	6.98	4.70	10.94	5.44	1.23
Wealth gini	162.03	206.74	229.95	43.46	16.64	0.91
Time (SO)	1.82	13.24	201.42	1.85	13.59	232.04

Note: Values represent accumulated (absolute) differences in the responses following a 15-std. shock to δ (over 300 periods) from the second-order solution with n_k gridpoints to the converged solution (n_k=500 and DEGM). Values are given as percent differences to steady state.

Impulse Responses: Histogram Method vs DEGM

- First order: IRFs converge to the same limit, but faster convergence with *DEGM*.
- Second order: The histogram method fails to approach the "true" solution as we increase the number of gridpoints. Particularly pronounced for inequality.

Comparison of ergodic moments

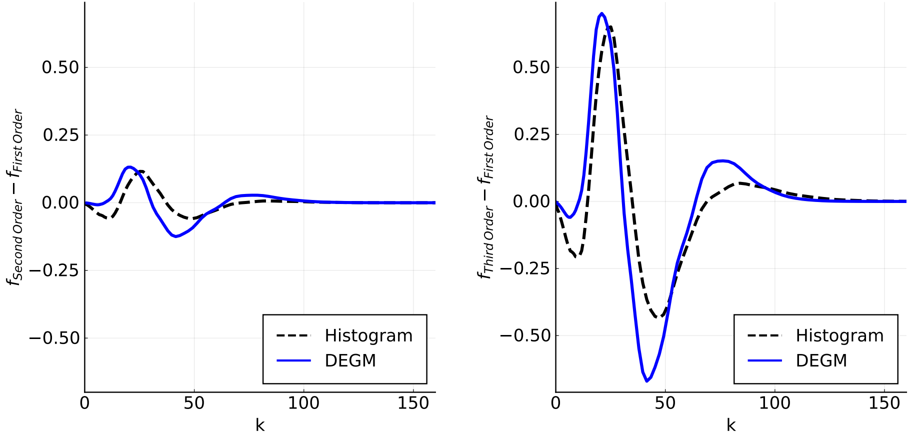
	Steady state		Second Order		Third Order	
Variable	Hstgrm	DEGM	Hstgrm	DEGM	Hstgrm	DEGM
Output	2.95	2.95	-0.2 (0.7)	-0.2 (0.7)	-0.3 (0.7)	-0.5 (0.7)
Capital stock	30.06	30.02	-0.5 (2.2)	-0.6 (2.1)	-1.0 (2.2)	-1.6 (2.1)
Wealth Gini	26.95	23.90	-1.4 (1.3)	2.1 (0.7)	1.8 (1.4)	12.4 (0.6)

Notes: Non-stochastic steady state levels across methods ($n_k = 100$, columns 1-2). Means and standard deviations (in brackets) across perturbation order and methods, in percent deviation from non-stochastic steady state (columns 3-6). Moments are averages of simulated data generated from pruned model dynamics (Andreasen et al., 2018) for $T = 10.000$ periods. Depreciation shocks are drawn from normal-inverse Gaussian distribution $F^{\nu}(0, \sigma_\delta, \tau_\delta)$.

Ergodic Moments: Implications of Aggregate Investment Risk

- As in Angeletos (2007), aggregate investment risk reduces the aggregate capital stock.
- Substitution effect dominates income effect. Strongest for less wealthy households who rely more on labor income.
- While a capital depreciation shock itself compresses the distribution of wealth, the risk of such a shock increases wealth inequality on average.
- The histogram method understates the impact of investment risk on economic activity and even more so on inequality.

Relative shift of the density of wealth with aggregate risk



Second Order Shift

Third Order Shift

Conclusion

Conclusion

- ***DEGM*** is a novel method for distributional dynamics that is fast, simple, shape preserving, and captures all nonlinearities.

- Key insight: Exploiting the monotonicity of the optimal policy and the CDF.

- Allows you to solve heterogeneous agent models with an order of magnitude smaller number of gridpoints and explore aggregate risk with higher order solutions.

- We find that aggregate investment risk has large effects on inequality. It increases wealth inequality by hollowing out the middle class.

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